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Equivalent Hamiltonians for *PT*-symmetric versions of dual 2D field theories

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Abstract
In the context of non-Hermitian but PT-symmetric Hamiltonians, we explore the systematics of the coefficients of the multiple commutators that appear in the perturbative expansions for Q, the exponent of the metric operator, and h, the equivalent Hermitian Hamiltonian. We find exact expressions for both Q and h for PT-symmetric versions of the dual two-dimensional quantum field theories, the sine-Gordon model and the massive Thirring model, and elucidate how these solutions arise from the structure of the multiple commutator coefficients.

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1. Introduction

Recent work on non-Hermitian systems has its origin in the seminal work of Bender and Boettcher [1] on the reality of the spectrum of the quantum-mechanical potential $\mathrm{i}x^N$ for $N \geqslant 2$. The reason behind this reality was subsequently realized to be the existence of an (unbroken) PT symmetry of the Hamiltonian.

However, a significant barrier to the physical interpretation of such theories was that the natural metric in Hilbert space was indefinite. With the discovery [2] that there was another metric, the CPT metric, which was indeed positive definite, this barrier was removed, although this metric is dynamically determined by the Hamiltonian itself and needs to be calculated in each case.

The mathematical basis for such theories was formalized by Mostafazadeh [3], with the role of PT symmetry generalized to the property of pseudo-Hermiticity (see equation (1)), and the metric η being recognized as generating a similarity transformation between the original pseudo-Hermitian Hamiltonian H and an equivalent Hermitian Hamiltonian h. In the case of PT-symmetric Hamiltonians, the role of η is played by PC. In [4], it was found convenient to write C in the form $C = \mathrm{e}^{Q}P$, where Q was a Hermitian operator satisfying $PQ = -QP$. Hence in this case $\eta = \mathrm{e}^{-Q}$, as in equation (2), which is indeed a positive-definite Hermitian operator.

In the majority of cases Q can only be calculated in some approximation scheme, notably perturbation theory, where successive terms in the expansion of Q have to be calculated by solving commutation relations arising from the condition of pseudo-Hermiticity. The first part of this contribution is concerned with finding the coefficients that occur in these multiple commutators and understanding how they arise. We do the same thing for the coefficients in the expansion of h.

In section 3, we give a rare example [5] of two related systems for which Q, and subsequently h, can be calculated exactly by summing up the entire perturbation series. Moreover, this is in quantum field theory rather than in quantum mechanics. The systems in question are PT-symmetric versions of the well-known dual field theories in (1+1) dimensions, the sine-Gordon model and the massive Thirring model.

In the appendix, we show how it is that these solutions arise given the structure of the perturbative coefficients that we found in section 1.

2. Commutation relations

We are concerned with Hamiltonians that are not Hermitian, but are PT-symmetric, and hence pseudo-Hermitian [3], i.e.

$$H^\dagger = \eta H \eta^{-1}. \tag{1}$$

Here η is Hermitian and positive definite, and is usefully written as

$$\eta = \mathrm{e}^{-Q}, \tag{2}$$

where Q is Hermitian. This is the operator first introduced in [4] as a useful tool in calculating the C operator.

Let us specialize to the case where $H = H_0 + \varepsilon H_1$, where H_0 is Hermitian, i.e. $H_0^\dagger = H_0$, and H_1 is anti-Hermitian, i.e. $H_1^\dagger = -H_1$.

2.1. Commutation relations for Q

We are looking for a perturbative solution, whereby

$$Q = \sum_r Q_r \varepsilon^r \quad (r \text{ odd}). \tag{3}$$

In the first place equation (1) becomes

$$\begin{aligned} H^\dagger &= \mathrm{e}^{-Q} H\, \mathrm{e}^{Q} \\ &= H + [H,Q] + \frac{1}{2!}[[H,Q],Q] + \frac{1}{3!}[[[H,Q],Q],Q] \\ &\quad + \cdots + \frac{1}{n!}\underbrace{[\ldots[H,Q],\ldots,Q]}_{n\text{ commutators}} + \cdots. \end{aligned} \tag{4}$$

Now insert $H = H_0 + \varepsilon H_1$, $Q = \sum_r Q_r \varepsilon^r$ and collect terms

$$\begin{aligned} &-2H_1 = [H_0, Q_1] \\ &0 = [H_0, Q_3] + \frac{1}{2!}[[H_1, Q_1], Q_1] + \frac{1}{3!}[[[H_0, Q_1], Q_1], Q_1] \\ &0 = [H_0, Q_5] + \frac{1}{2!}([[H_1, Q_1], Q_3] + [[H_1, Q_3], Q_1]) + \frac{1}{3!}([[[H_0, Q_1], Q_1], Q_3] + \text{perms}) \\ &\qquad + \frac{1}{4!}[[[[H_1, Q_1], Q_1], Q_1], Q_1] + \frac{1}{5!}[[[[[H_0, Q_1], Q_1], Q_1], Q_1], Q_1] \\ &\qquad \ldots. \end{aligned} \tag{5}$$

Here the coefficients are simple: just $1/n!$ for each n-fold commutator. But in each equation we can use the previous equations to eliminate previously calculated $[H_0, Q_r]$. The equations then become

$$
\begin{aligned}
&[H_0, Q_1] = -2H_1,\\
&[H_0, Q_3] = -\tfrac{1}{6}[[H_1, Q_1], Q_1],\\
&[H_0, Q_5] = -\tfrac{1}{6}\left([[H_1, Q_1], Q_3] + [[H_1, Q_3], Q_1]\right) + \tfrac{1}{360}[[[[H_1, Q_1], Q_1], Q_1], Q_1]\\
&[H_0, Q_7] = -\tfrac{1}{6}[[H_1, Q_3], Q_3] - \tfrac{1}{6}([[H_1, Q_1], Q_5] + [[H_1, Q_5], Q_1])\\
&\qquad + \tfrac{1}{360}([[[[H_1, Q_1], Q_1], Q_1], Q_3] + \text{perms})\\
&\qquad - \tfrac{1}{15{,}120}[[[[[[H_1, Q_1], Q_1], Q_1], Q_1], Q_1], Q_1],\\
&\qquad \ldots,
\end{aligned}
\tag{6}
$$

with coefficients $-1/6$, $1/360$, $-1/15,120$, etc.

The fundamental question to ask is: what are these coefficients, and what is the general coefficient?

By inspection of these and higher-order coefficients, it is possible to recognize c_n, the coefficient of the $2n$-fold commutator, as

$$
\begin{aligned}
c_n &= -\frac{2B_{2n}}{(2n)!}\\
&= \text{coefficient of } z^{2n} \text{ in } -z\coth\tfrac{1}{2}z.
\end{aligned}
\tag{7}
$$

The supplementary question is why these coefficients should have anything to do with a hyperbolic cotangent. The answer lies in the recursion relation whereby the c_n are built up, namely

$$
c_n = -\frac{1}{(2n)!} - \sum_{r=1}^{n} \frac{c_{n-r}}{(2r+1)!},
\tag{8}
$$

where the sum arises from previous H_0 commutators.

This happens to be the same as one of the recursion relations one can write for the coefficients c_n in

$$
-z\coth\tfrac{1}{2}z = \sum_{n=0}^{\infty} c_n z^{2n}.
\tag{9}
$$

Thus, multiplying both sides of equation (9) by $e^{z/2}$ we obtain

$$
-z(e^z + 1) = \sum_{n=0}^{\infty} c_n z^{2n}(e^z - 1),
\tag{10}
$$

i.e.

$$
-z\left(1 + \sum_{n=0}^{\infty} \frac{z^n}{n!}\right) = \sum_{m=0}^{\infty} c_m z^{2m} \sum_{s=1}^{\infty} \frac{z^s}{s!}.
\tag{11}
$$

Equating coefficients of z^{2n+1} gives precisely equation (8).

2.2. *Commutation relations for h*

The equivalent Hermitian Hamiltonian h is given by

$$
h = e^{-Q/2} H\, e^{Q/2}
$$

$$= H + \frac{1}{2}[H, Q] + \frac{1}{4\times 2!}[[H, Q], Q] + \frac{1}{8\times 3!}[[[H, Q], Q], Q] + \cdots + \frac{1}{2^n \times n!}\underbrace{[\ldots[H, Q], \ldots, Q]}_{n \text{ commutators}} + \cdots. \qquad (12)$$

Inserting $H = H_0 + \varepsilon H_1$, $Q = \sum_r Q_r \varepsilon^r$, $h = \sum_{r\ \mathrm{even}} h_r \varepsilon^r$ into these relations, the first few equations are

$$\begin{aligned}
h_0 &= H_0\\
h_2 &= \frac{1}{2}[H_1, Q_1] + \frac{1}{4\times 2!}[[H_0, Q_1], Q_1]\\
h_4 &= \frac{1}{2}[H_1, Q_3] + \frac{1}{4\times 2!}([[H_0, Q_1], Q_3] + [[H_0, Q_3], Q_1]) + \frac{1}{8\times 3!}[[[H_1, Q_1], Q_1], Q_1]\\
&\quad + \frac{1}{16\times 4!}[[[[H_0, Q_1], Q_1], Q_1], Q_1]\\
h_6 &= \frac{1}{2}[H_1, Q_5] + \frac{1}{4\times 2!}([[H_0, Q_1], Q_5] + [[H_0, Q_5], Q_1]) + \frac{1}{4\times 2!}[[H_0, Q_3], Q_3]\\
&\quad + \frac{1}{8\times 3!}([[[H_1, Q_1], Q_1], Q_3] + \text{perms})\\
&\quad + \frac{1}{16\times 4!}([[[[H_0, Q_1], Q_1], Q_3], Q_1] + \text{perms})\\
&\quad + \frac{1}{32\times 5!}[[[[[H_1, Q_1], Q_1], Q_1], Q_1], Q_1]\\
&\quad + \frac{1}{64\times 6!}[[[[[[H_0, Q_1], Q_1], Q_1], Q_1], Q_1], Q_1]. \qquad (13)
\end{aligned}$$

Again, the coefficients are simple: just $1/(2^n n!)$. But eliminating $[H_0, Q_r]$ gives

$$\begin{aligned}
h_0 &= H_0\\
h_2 &= \frac{1}{4}[H_1, Q_1]\\
h_4 &= \frac{1}{4}[H_1, Q_3] - \frac{1}{192}[[[H_1, Q_1], Q_1], Q_1]\\
h_6 &= \frac{1}{4}[H_1, Q_5] - \frac{1}{192}([[[H_1, Q_1], Q_1], Q_3] + \text{perms})\\
&\quad + \frac{1}{7680}[[[[[H_1, Q_1], Q_1], Q_1], Q_1], Q_1]\\
&\ldots. \qquad (14)
\end{aligned}$$

Once again we can ask: what are these coefficients and what is the general coefficient? It turns out that a_n, the coefficient of the $(2n-1)$-fold commutator, is

$$\begin{aligned}
a_n &= -\frac{E_{2n-1}(0)}{2^{2n-1}\times(2n-1)!},\\
&= \text{coefficient of } z^{2n-1} \text{ in } \tanh(z/4). \qquad (15)
\end{aligned}$$

The reason again lies in the recursion relation generating the a_n, namely

$$a_n = \frac{1}{2^{2n-1}\times(2n-1)!} + \sum_{r=1}^{n} \frac{c_{n-r}}{2^{2r}\times(2r)!}, \qquad (16)$$

which is the same as one of the recursion relations for the coefficients in

$$\tanh(z/4) = \sum_{n=1}^{\infty} a_n z^{2n-1}. \tag{17}$$

Thus,

$$\begin{aligned}(z\coth(z/2))(\cosh(z/2)-1) &= -\left(\sum_{m=0}^{\infty} c_m z^{2m}\right)\left(\sum_{r=1}^{\infty} \frac{z^{2r}}{2^{2r}\times(2r)!}\right)\\ &= -\sum_{n=1}^{\infty} z^{2n} \sum_{r=1}^{n} \frac{c_{n-r}}{2^{2r}\times(2r)!}\end{aligned} \tag{18}$$

and

$$z\sinh(z/2) = \sum_{n=1}^{\infty} \frac{z^{2n}}{2^{2n-1}\times(2n-1)!}. \tag{19}$$

Subtracting gives

$$\begin{aligned}\sum_{n=1}^{\infty} a_n z^{2n} &= z[\sinh(z/2) - \coth(z/2)(\cosh(z/2)-1)]\\ &= z\tanh(z/4).\end{aligned} \tag{20}$$

3. *PT*-symmetric versions of the sine-Gordon and massive Thirring models

3.1. Modified sine-Gordon model

The conventional sine-Gordon model, in (1+1) dimensions, has

$$\mathcal{L}_{\mathrm{SG}} = \frac{1}{2}(\partial_\mu\varphi)^2 + \frac{M^2}{\lambda^2}(\cos\lambda\varphi - 1), \tag{21}$$

or correspondingly

$$\mathcal{H}_{\mathrm{SG}} = \frac{1}{2}\Pi^2 + \frac{1}{2}(\nabla\varphi)^2 + \frac{M^2}{\lambda^2}(1-\cos\lambda\varphi). \tag{22}$$

As a modified version, which is PT-symmetric but not Hermitian, we consider

$$\mathcal{H} = \frac{1}{2}\Pi^2 + \frac{1}{2}(\nabla\varphi)^2 + \frac{M^2}{\lambda^2}(1-\cos\lambda\varphi - \mathrm{i}\varepsilon\sin\lambda\varphi). \tag{23}$$

So in this case $\mathcal{H}_0 = \mathcal{H}_{\mathrm{SG}}$ and $\mathcal{H}_1 = -\mathrm{i}(M^2/\lambda^2)\sin\lambda\varphi$.

The first few equations for the Q_n are

$$\begin{aligned}&[H_0, Q_1] = -2H_1,\\ &[H_0, Q_3] = -\tfrac{1}{6}[[H_1, Q_1], Q_1],\\ &[H_0, Q_5] = -\tfrac{1}{6}([[H_1, Q_1], Q_3] + [[H_1, Q_3], Q_1]) + \tfrac{1}{360}[[[[H_1, Q_1], Q_1], Q_1], Q_1].\end{aligned} \tag{24}$$

It is relatively easy to spot an ansatz for Q_1 which will produce the desired result, namely

$$Q_1 = \xi_1 \int_x \Pi_x. \tag{25}$$

Then

$$[H_0, Q_1] = \xi_1 \int_{xy} \left[\frac{1}{2}\partial_1\varphi_x\partial_1\varphi_x - \frac{M^2}{\lambda^2}\cos\lambda\varphi, \Pi_y \right]$$

$$= \mathrm{i}\xi_1 \int_x \left(\underbrace{\partial_1^2\varphi_x}_{\int\to 0} + \underbrace{\frac{M^2}{\lambda}\sin\lambda\varphi}_{\propto\mathcal{H}_1} \right), \tag{26}$$

by virtue of the equal-time canonical commutation relation $[\varphi_x, \Pi_y] = \mathrm{i}\delta_{xy}$. As indicated, the first term integrates to zero and the second is proportional to $\mathcal{H}_1$. So the first of equations (24) is satisfied provided that $\xi_1 = 2/\lambda$.

Similarly, if we set $Q_r = \xi_r \int_x \Pi_x$ in general, the second equation is satisfied with $\xi_3 = \xi_1/3$ and the third with $\xi_5 = \xi_1/5$.

We thus seem to be generating the series for $\tanh^{-1}\varepsilon$, giving the all-orders result

$$Q = \frac{2\delta}{\lambda}\int_x \Pi_x, \tag{27}$$

where $\delta = \tanh^{-1}\varepsilon$. Note that this result only makes sense for $|\varepsilon| < 1$.

We can verify the result *a posteriori* by constructing

$$h = \mathrm{e}^{-(\delta/\lambda)\int_x \Pi_x} H\, \mathrm{e}^{(\delta/\lambda)\int_x \Pi_x}. \tag{28}$$

This just shifts φ: $\varphi \to \varphi + \mathrm{i}\delta/\lambda$. Thus

$$\cos\lambda\varphi + \mathrm{i}\varepsilon\sin\lambda\varphi \equiv \operatorname{sech}\delta\cos(\lambda\varphi - \mathrm{i}\delta)$$
$$\to \operatorname{sech}\delta\cos\lambda\varphi. \tag{29}$$

The equivalent Hermitian Hamiltonian h thus turns out to be the sine-Gordon model again, but with bare mass M' given by

$$(M')^2 = M^2\operatorname{sech}\delta = M^2(1-\varepsilon^2)^{\frac{1}{2}}. \tag{30}$$

We can now understand the restriction $|\varepsilon| < 1$: the PT symmetry is spontaneously broken for $|\varepsilon| > 1$, when M' becomes pure imaginary.

3.2. Modified massive Thirring model

The conventional massive Thirring model, again in (1+1) dimensions, has

$$\mathcal{L}_{\mathrm{MT}} = \bar{\psi}(\mathrm{i}\partial\!\!\!/ - m)\psi + \tfrac{1}{2}g(\bar{\psi}\gamma^\mu\psi)(\bar{\psi}\gamma_\mu\psi), \tag{31}$$

or correspondingly

$$\mathcal{H}_{\mathrm{MT}} = \bar{\psi}(-\mathrm{i}\nabla\!\!\!\!/ + m)\psi - \tfrac{1}{2}g(\bar{\psi}\gamma^\mu\psi)(\bar{\psi}\gamma_\mu\psi). \tag{32}$$

This is equivalent to the sine-Gordon model, with the correspondence

$$\frac{\lambda^2}{4\pi} = \frac{1}{1-g/\pi}, \qquad M^2 = m\Lambda, \tag{33}$$

where Λ is a renormalization scale. In particular $\lambda = \sqrt{4\pi} \leftrightarrow g = 0$, the free fermion theory.

Our modified version involves a 'γ_5-dependent mass':

$$\mathcal{H} = \bar{\psi}(-\mathrm{i}\nabla\!\!\!\!/ + m(1+\varepsilon\gamma_5))\psi - \tfrac{1}{2}g(\bar{\psi}\gamma^\mu\psi)(\bar{\psi}\gamma_\mu\psi). \tag{34}$$

Here, $\gamma_0 = \sigma_1, \gamma_1 = \mathrm{i}\sigma_2, \gamma_5 \equiv \gamma_0\gamma_1 = -\sigma_3$, so the additional term is again PT-symmetric but non-Hermitian.

First consider $g = 0(\lambda = \sqrt{4\pi})$ and write

$$Q_1 = \int_{xy} \psi_x^\dagger (G_1)_{xy} \psi_y, \qquad H_0 = \int_{xy} \psi_x^\dagger D_{xy} \psi_y, \tag{35}$$

where $D = \gamma_0(-\mathrm{i}\not\nabla + m) = -\mathrm{i}\gamma_5\partial_1 + m\gamma_0$.

Using the canonical equal-time anti-commutation relations $\{\psi_x^\dagger, \psi_y\} = \delta_{xy}$, the equation $[H_0, Q_1] = -2H_1$ reads

$$\begin{aligned} -2m \int \bar\psi\gamma_5\psi &= \int [\psi^\dagger D\psi, \psi^\dagger G_1\psi] \\ &= \int \psi^\dagger [D, G_1]\psi \\ &= \int \psi^\dagger[-\mathrm{i}\gamma_5\partial_1 + m\gamma_0, G_1]\psi, \end{aligned} \tag{36}$$

of which a particular solution is $G_1 = -\gamma_5$.

Similarly, if in general we set $G_r = -\xi_r\gamma_5$, then equations (24) give $\xi_3 = 1/3$ and $\xi_5 = 1/5$. Again, we seem to be generating the series for $\tanh^{-1}\varepsilon$, with the all-orders result

$$Q = -\delta \int_x (\psi^\dagger\gamma_5\psi)_x, \tag{37}$$

where $\delta = \tanh^{-1}\varepsilon$, as before.

We can check this result by constructing

$$h = \exp\left(\frac{1}{2}\delta \int_x (\psi^\dagger\gamma_5\psi)_x\right) H \exp\left(-\frac{1}{2}\delta \int_x (\psi^\dagger\gamma_5\psi)_x\right). \tag{38}$$

By virtue of the Lorentz-like commutation relations

$$[\gamma_5, \gamma_0] = 2\gamma_1, \qquad [\gamma_5, \gamma_1] = 2\gamma_0, \tag{39}$$

this is just

$$h = \bar\psi(-\mathrm{i}\not\nabla + \mu)\psi, \tag{40}$$

where $\mu = m \operatorname{sech}\delta = m(1-\varepsilon^2)^{\frac{1}{2}}$, in agreement with (30).

It is important to note that this Q also works for $g \neq 0$, since

$$(\bar\psi\gamma^\mu\psi)(\bar\psi\gamma_\mu\psi) = (\psi^\dagger\psi)^2 - (\psi^\dagger\gamma_5\psi)^2. \tag{41}$$

Each term on the right-hand side commutes with $Q = -\delta\int_x(\psi^\dagger\gamma_5\psi)_x$.

Thus, the only effect on H is to change the γ_5-dependent mass term $m\bar\psi(1+\varepsilon\gamma_5)\psi$ to a normal mass term $\mu\bar\psi\psi$. Again, the PT symmetry is broken if $|\varepsilon| > 1$ because μ becomes pure imaginary.

3.2.1. Another solution. As usual, Q is not unique, as has been discussed in [6–8]. At any stage in the solution of the commutation relations (6) for Q_r, we can add to Q_r anything that commutes with H_0. In the present case, most of the other solutions would be unsatisfactory for one reason or another. We illustrate the point by presenting an interesting alternative solution, namely

$$G = \mathrm{i}\gamma_5 \tanh^{-1}\left(\frac{\varepsilon m}{\not\nabla}\right). \tag{42}$$

This mixes the kinetic term and the γ_5 mass term. In the free-field case $g = 0$ the corresponding h is

$$h = \bar\psi\left[m - \mathrm{i}\not\nabla\left(1 + \frac{\varepsilon^2 m^2}{\partial^2}\right)^{\frac{1}{2}}\right]\psi, \tag{43}$$

which is a rather perverse way of writing the free-field theory with mass μ! Thus the equation of motion is

$$\left[\partial_0^2 - \partial^2\left(1 + \frac{\varepsilon^2 m^2}{\partial^2}\right)\right]\psi = -m^2\psi, \tag{44}$$

i.e.

$$\partial^2\psi = -\underbrace{m^2(1-\varepsilon^2)}_{\mu^2}\psi. \tag{45}$$

This Q is unsatisfactory because it is non-local and moreover only defined in a limited range of p-space.

4. Conclusions

In this contribution, we have identified the coefficients of the multiple commutators involved in the perturbative expansions for the Q operator, determined by

$$H^\dagger = \mathrm{e}^{-Q} H\, \mathrm{e}^{Q}, \tag{46}$$

and for the equivalent Hermitian Hamiltonian h, then given by

$$h = \mathrm{e}^{-\frac{1}{2}Q} H\, \mathrm{e}^{\frac{1}{2}Q}. \tag{47}$$

In the context of (1+1)-dimensional field theory, we have found exact expressions for Q and h in PT-symmetric versions of the sine-Gordon and massive Thirring models. The appendix shows how these solutions can be understood in terms of the properties of the above coefficients.

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Appendix. Relation to the commutator coefficients

In this appendix, we elucidate how the functions of section 3, the arctanh and the sech, arise from the coefficients we found in section 2, which generate coth and tanh, respectively.

A.1. Structure of Q

The important feature to note here is that the Q_r of section 3 all have the same structure, $Q_r = \alpha_r Q_1$, so that $Q = \left(\sum_r^\infty \alpha_r \varepsilon^r\right) Q_1$.

Therefore, from

$$[H_0, Q_1] = c_0 H_1, \qquad [H_0, Q_3] = c_1[[H_1, Q_1], Q_1], \tag{A.1}$$

we get

$$[[H_1, Q_1], Q_1] = \alpha_3 \frac{c_0}{c_1} H_1 = 4H_1 \tag{A.2}$$

(recall that $\alpha_3 = 1/3$, $c_0 = -2$, $c_1 = -1/6$).

Then the equation

$$[H_0, Q_5] = c_1([[H_1, Q_1], Q_3] + [[H_1, Q_3], Q_1]) + c_2[[[[H_1, Q_1], Q_1], Q_1], Q_1] \tag{A.3}$$

becomes

$$-2\alpha_5 H_1 = \left[2(4c_1)\alpha_1\alpha_3 + (16c_2)\alpha_1^4\right] H_1, \tag{A.4}$$

which gives $\alpha_5 = 1/5$.

In fact all the equations can be generated by the expansion of

$$-2\sum_{r\,odd}^{\infty} \alpha_r \varepsilon^r = \varepsilon \sum_{r=0}^{\infty} \underbrace{4^r c_r}_{\hat{c}_r} (\alpha_1\varepsilon + \alpha_3\varepsilon^3 + \alpha_5\varepsilon^5 + \cdots)^{2r}, \tag{A.5}$$

namely

$$\begin{aligned}
-2\alpha_1 &= \hat{c}_0 \\
-2\alpha_3 &= \hat{c}_1\alpha_1^2 \\
-2\alpha_5 &= 2\hat{c}_1\alpha_1\alpha_3 + \hat{c}_2\alpha_1^4 \\
-2\alpha_7 &= \hat{c}_1\left(2\alpha_1\alpha_5 + \alpha_3^2\right) + 4\hat{c}_2\alpha_1^3\alpha_3 + \hat{c}_3\alpha_1^6 \\
&\ldots .
\end{aligned} \tag{A.6}$$

Here, the multiplicities are the number of permutations of a given commutator structure.

Then, assuming that α_r are the coefficients of arctanh, equation (A.5) reads

$$\begin{aligned}
-2\delta &= \varepsilon \sum_r^{\infty} c_r (2\delta)^{2r} \\
&= \tanh\delta(-2\delta\coth\delta),
\end{aligned} \tag{A.7}$$

which is indeed consistent.

A.2. Structure of h

With $[[H_1, Q_1], Q_1] = 4H_1$ and $[H_1, Q_1] = -2H_0$, the equations for h, namely

$$\begin{aligned}
h_2 &= a_1[H_1, Q_1], \\
h_4 &= a_1[H_1, Q_3] + a_2[[[H_1, Q_1], Q_1], Q_1], \\
h_6 &= a_1[H_1, Q_5] + a_2([[[H_1, Q_1], Q_1], Q_3] + \text{perms}) + a_3[[[[[H_1, Q_1], Q_1], Q_1], Q_1], Q_1],
\end{aligned} \tag{A.8}$$

become

$$\begin{aligned}
h_2 &= -2\hat{a}_1\alpha_1 H_0, \\
h_4 &= -2\left(\hat{a}_1\alpha_3 + \hat{a}_2\alpha_1^3\right) H_0, \\
h_6 &= -2\left(\hat{a}_1\alpha_5 + 3\hat{a}_2\alpha_1^2\alpha_3 + \hat{a}_3\alpha_1^5\right) H_0,
\end{aligned} \tag{A.9}$$

where $\hat{a}_r = 4^{r-1}a_r$.

Again, $h \equiv \sum h_r\varepsilon^r$ is generated by

$$\begin{aligned}
1 - 2\varepsilon \sum_{r=1}^{\infty} &\hat{a}_r (\alpha_1\varepsilon + \alpha_3\varepsilon^3 + \alpha_5\varepsilon^5 + \cdots)^{2r-1} \\
&= 1 - \varepsilon \sum_{r=1}^{\infty} a_r (2\delta)^{2r-1} \\
&= 1 - \tanh\delta \times \tanh(\delta/2) \\
&= \operatorname{sech}\delta.
\end{aligned} \tag{A.10}$$

References

[1] Bender C M and Boettcher S 1998 *Phys. Rev. Lett.* **80** 5243
Bender C M, Boettcher S and Meisinger P N 1999 *J. Math. Phys.* **40** 2201

[2] Bender C M, Brody D C and Jones H F 2002 *Phys. Rev. Lett.* **89** 270401
Bender C M, Brody D C and Jones H F 2004 *Phys. Rev. Lett.* **92** 119902(E)

[3] Mostafazadeh A 2002 *J. Math. Phys.* **43** 205
Mostafazadeh A 2003 *J. Phys. A: Math. Gen.* **36** 7081

[4] Bender C M, Brody D C and Jones H F 2004 *Phys. Rev.* D **70** 025001
Bender C M, Brody D C and Jones H F 2005 *Phys. Rev.* D **71** 049901(E)

[5] Bender C M, Jones H F and Rivers R J 2005 *Phys. Lett.* B **625** 333

[6] Mostafazadeh A 2003 *J. Math. Phys.* **44** 974

[7] Geyer H B, Scholtz F G and Snyman I 2004 *Czech. J. Phys.* **54** 1069

[8] Jones H F 2005 *J. Phys. A: Math. Gen.* **38** 1741